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ON A LINEAR DIFFERENTIAL GAME OF ENCOUNTER

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S. I. TARLINSKII
(Sverdlovsk)

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We examine the problem of the encounter of conflict-controlled objects with a specified set at a fixed instant of time. We present one condition under which such a problem has a solution. The results are closely related to those in [1-7].

1. We consider a dynamic system of two controlled objects. One of them, subject to the first player, is described by the linear differential equation

$$dy/dt = A^{(1)}(t)\,y + B^{(1)}(t)\,u + f^{(1)}(t) \tag{1.1}$$

where y is an $n^{(1)}$-dimensional vector. The other object, subject to the second player, is described by the equation

$$dz/dt = A^{(2)}(t)\,z + B^{(2)}(t)\,v + f^{(2)}(t) \tag{1.2}$$

Here z is an $n^{(2)}$-dimensional vector. The game is played on a specified time interval $[t_0, \vartheta]$. At each instant the players' controls are constrained by

$$u[t] \in P, \quad v[t] \in Q \tag{1.3}$$

where P, Q are convex compacta in the appropriate vector spaces.

By $\{p\}_m$ we denote the vector comprised of the first m coordinates of a vector p and we let $\rho(q, N)$ denote the Euclidean distance from a point q to a set N. By the game's hypotheses, in the phase space $\{x\}_m = \{z - y\}_m$ $(m \leqslant n^{(1)},\ m \leqslant n^{(2)})$ a convex compact set M is specified and an initial game position $\{t_0, y_0, z_0\}$, $y[t_0] = y_0$, $z[t_0] = z_0$, is fixed. The first player, dealing with control u, strives to minimize at instant ϑ the quantity

$$\gamma = \rho(\{z - y\}_m,\ M) \tag{1.4}$$

The second player, by choosing control v, strives to maximize the value of the payoff γ in (1.4) at this same instant ϑ.

Let us make the problem statement more precise. By the first player's position strategy we mean a function $U = U(t, y, z)$ which associates a convex compact set U with every game position $\{t, y, z\}$ where $U \subset P$. Any integrable realization $v = v[t]$ $(t_0 \leqslant t \leqslant \vartheta)$, constrained by the condition $v[t] \in Q$, is admissible for the second player, moreover, this realization $v[t]$ can be produced on the basis of an arbitrary control law using any conceivable information on the course of the process. Let Δ be some partitioning of the interval $[t_0, \vartheta]$ into a finite number of parts by the points τ_i $(i = 0, 1, \ldots)$. Any absolutely continuous function $y_\Delta[t] = y_\Delta[t;\ t_0,\ y_0,\ U]$ satisfying the relation

$$dy_\Delta/dt = A^{(1)}(t)\,y_\Delta + B^{(1)}(t)\,u[\tau_i] + f^{(1)}(t) \tag{1.5}$$

for almost all values of $t \in [\tau_i, \tau_{i+1}]$ is called an approximate motion of system (1.1). Here $u[\tau_i] \in U(\tau_i, y_\Delta[\tau_i], z_\Delta[\tau_i])$, where the motion $z[t] = z[t; t_0, z_0, v_\Delta[\cdot]]$ is generated by the integrable function $v_\Delta[\cdot] = v_\Delta[t]$ and satisfies Eq. (1.2) almost everywhere. An absolutely continuous function $y[t] = y[t; t_0, y_0, U]$ is called the motion of system (1.1), generated by the strategy U and by the initial conditions $y[t_0] = y_0$, if there exists a sequence of approximate motions $y_{\Delta_k}[t]$ of (1.5), converging uniformly to $y[t]$, i.e.

$$\lim_{k\to\infty} y_{\Delta_k}[t] = y[t], \qquad \lim_{k\to\infty} \sigma(\Delta_k) = 0$$

$$\sigma(\Delta) = \max_i (\tau_{i+1} - \tau_i)$$

The generating motions $z[t] = z[t; t_0, z_0, v_{\Delta_k}[\cdot]]$ of the function $v_{\Delta_k}[t]$ can vary arbitrarily as Δ_k changes, just as long as the sequence $v_{\Delta_k}[t]$ converges weakly to some measurable function $v[t]$. Using the results of [5] we can show that the family of such motions is a nonempty set compact in itself. We pose the following problem [5].

Problem 1.1. Let a final instant ϑ be given and a target set M specified. We are required to find the first player's optimal minimax strategy $U_0 = U_0(t, y, z)$ satisfying the condition

$$\{\rho(\{z[\vartheta] - y[\vartheta]\}_m, M) \mid X[t_0, y_0, z_0, U_0]\} \leqslant$$
$$\min_U \max_{y[\cdot], z[\cdot]} \{\rho(\{z[\vartheta] - y[\vartheta]\}_m M) \mid X[t_0, y_0, z_0, U]\}$$

Here the symbol $X[t_0, y_0, z_0, U]$ denotes the family of all motions

$$y[\cdot] = y[t; t_0, y_0, U], \ z[\cdot] = z[t; t_0, z_0, v[\cdot]], \ v[\cdot] \in Q$$

of systems (1.1), (1.2), corresponding to the initial position $y[t_0] = y_0$, $z[t_0] = z_0$.

2. The extremal construction introduced in [1, 5, 6] is the foundation for solving Problem 1.1. For completeness of presentation we describe the fundamental elements of this construction. By $Y[t, \tau]$, $Z[t, \tau]$ we denote the fundamental matrices of the following equations: $dy/dt = A^{(1)}(t)\, y, \qquad dz/dt = A^{(2)}(t)\, z$

For each vector $\{t, y, z\}$ and for an m-dimensional vector l we define the function

$$\varphi(t, y, z, l) = l' x_0(t, y, z) - \int_t^\vartheta \rho_P(\xi, l)\, d\xi + \int_t^\vartheta \rho_Q(\xi, l)\, d\xi \tag{2.1}$$

Here the prime denotes transposition and the quantities $x_0(t, y, z)$, $\rho_P(t, l)$, $\rho_Q(t, l)$, $\rho_M(l)$ are given by the relations

$$\rho_P(t, l) = \max_{u\in P} l' \{Y[\vartheta, t] B^{(1)}(t) u\}_m \tag{2.2}$$

$$\rho_Q(t, l) = \max_{v\in Q} l' \{Z[\vartheta, t] B^{(2)}(t) v\}_m \tag{2.3}$$

$$\rho_M(l) = \max_{m\in M} l'm$$

$$x_0(t, y, z) = \{Z[\vartheta, t] z - Y[\vartheta, t] y\}_m +$$
$$\left\{\int_t^\vartheta Z[\vartheta, \xi] f^{(2)}(\xi)\, d\xi - \int_t^\vartheta Y[\vartheta, \xi] f^{(1)}(\xi)\, d\xi\right\}_m$$

We now set

$$\varepsilon(t, y, z) = \max_{\|l\|=1} \varphi(t, y, z, l) \tag{2.4}$$

Note that in the region $\varepsilon(t, y, z) > 0$ the quantity $\varepsilon(t, y, z)$ defines the program maximin distance of the phase point $\{x(\vartheta)\}_m = \{z(\vartheta) - y(\vartheta)\}_m$ from set M if the auxiliary program game had started from the position $y(t) = y$, $z(t) = z$ (see [5]). For each vector $\{t, y, z\}$ we specify a set $L_0(t, y, z)$ of m-dimensional vectors l_0.

$$L_0(t, y, z) = \{l_0 : \|l_0\| = 1, \ \varphi(t, y, z, l_0) = \varepsilon(t, y, z)\} \tag{2.5}$$

where the quantities $\varphi(t, y, z, l_0)$, $\varepsilon(t, y, z)$ have been defined by formulas (2.1), (2.4). Clearly, each of the sets $L_0(t, y, z)$ is closed and bounded. Let us assume that the following condition is fulfilled.

Condition 2.1. In the region $\varepsilon(t, y, z) > 0$ we can specify, for each vector $v^* \in Q$, a vector $u^* \in P$ such that the inequality

$$\psi(t, u^*, v^*, l_0) \leqslant 0 \tag{2.6}$$

$$\psi(t, u, v, l) = \rho_P(t, l) - \rho_Q(t, l) + l'\{Z[\vartheta, t] B^{(2)}(t) v\}_m - l'\{Y[\vartheta, t] B^{(1)}(t) u\}_m \tag{2.7}$$

where the quantities $\rho_P(t, l)$, $\rho_Q(t, l)$ are defined by conditions (2.2), (2.3), is valid for all vectors $l_0 \in L_0(t, y, z)$ simultaneously.

We specify a set $H(t, y, z)$ in the m-dimensional space $\{h\}_m$. We say that $h \in H(t, y, z)$ if and only if the inequality

$$\rho_Q(t, l_0) - \rho_P(t, l_0) \geqslant l_0' h \tag{2.8}$$

is valid for all vectors $l_0 \in L_0(t, y, z)$ of (2.5). Obviously, the sets $H(t, y, z)$ are convex and closed. Let us prove the following lemma.

Lemma 2.1. For the fulfillment of Condition 2.1 it is necessary and sufficient that each of the sets $H(t, y, z)$ be nonempty and that the inequality

$$\max_{h \in H} s'h \geqslant \rho_Q(t, s) - \rho_P(t, s), \quad H = H(t, y, z) \tag{2.9}$$

hold for any m-dimensional vector s.

Indeed, suppose that Condition 2.1 is valid. Consequently, for any vector $v^* \in Q$ we can find a vector $u^* \in P$ satisfying inequality (2.6). Then from the definition of sets $H(t, y, z)$ we obtain

$$\{Z(\vartheta, t) B^{(2)}(t) v^*\}_m - \{Y(\vartheta, t) B^{(1)}(t) u^*\}_m \in H(t, y, z)$$

Therefore, the inequality

$$\max_{h \in H} s'h \geqslant s'\{Z(\vartheta, t) B^{(2)}(t) v^*\}_m - s'\{Y(\vartheta, t) B^{(1)}(t) u^*\}_m \tag{2.10}$$

is fulfilled for any vector s. From this and from (2.2) we have

$$\max_{h \in H} s'h \geqslant s'\{Z(\vartheta, t) B^{(2)}(t) v^*\}_m - \rho_P(t, s) \tag{2.11}$$

Since relation (2.11) is valid for any vector $v^* \in Q$, we finally obtain inequality (2.9) from (2.11), (2.3).

Conversely, suppose that (2.9) is fulfilled. We specify an arbitrary vector $v^* \in Q$. Then relation (2.11) follows immediately from (2.9). We introduce a convex compact set

$$F(t, v) = \{\{Z(\vartheta, t) B^{(2)}(t) v - Y(\vartheta, t) B^{(1)}(t) u\}_m : u \in P\} \tag{2.12}$$

From the theorem on the separability of convex sets [8] and from (2.11) it follows that the intersection of the sets $H(t, y, z)$ from (2.9) and $F(t, v^*)$ from (2.12) is not empty. Therefore, there exists a vector $u^* \in P$ for which inequality (2.10) is fulfilled and, consequently, inequality (2.6), which completes the proof of Lemma 2.1.

3. Let us present the proof of an auxiliary assertion with whose help Problem 1.1 is solved. In the space of vectors $\{t, y, z\}$ we specify a bounded set N. For a preassigned sufficiently-small positive number $\beta > 0$ we define the set

$$\Gamma_\beta = \{t, y, z : \{t, y, z\} \in N,\ t \in [t_0, \vartheta],\ \varepsilon(t, y, z) \geqslant \beta\} \tag{3.1}$$

Now let

$$L(t_*, y_*, z_*, \eta) = \bigcup_{\{t, y, z\}} L_0(t, y, z) \tag{3.2}$$

$$(|t - t_*| \leqslant \eta, \|y - y_*\| \leqslant \eta, \|z - z_*\| \leqslant \eta)$$

where the sets $L_0(t, y, z)$ have been defined by formula (2.5). The following assertion is valid.

Lemma 3.1. Let Condition 2.1 be fulfilled. Then, for any number $\alpha > 0$ we can find a number $\eta > 0$ such that the estimate

$$\max_{v \in Q} \min_{u \in P} \max_{l \in L} \psi(t_*, u, v, l) \leqslant \alpha \tag{3.3}$$

holds, where $L = L(t_*, y_*, z_*, \eta)$ is from (3.2) and the function $\psi(t, u, v, l)$ is defined by relation (2.7). Moreover, estimate (3.3) may be made uniform with respect to all $\{t_*, y_*, z_*\} \in \Gamma_\beta$.

The proof of Lemma 3.1 follows from Condition 2.1 as well as from the continuity of the function $\psi(t, u, v, l)$ in (2.7) and from the semicontinuity with respect to $\{t, y, z\}$ of the sets $L_0(t, y, z)$ in (2.5).

By $y_\Delta[t] = y_\Delta[t; t_*, y_*, u]$, $z_\Delta[t] = z_\Delta[t; t_*, z_*, v]$ we denote the motions of systems (1.1), (1.2), generated on the interval $[t_*, t_* + \Delta]$ by the controls $u \in P$, $v \in Q$ and by the initial conditions $y_\Delta[t_*] = y_*$, $z_\Delta[t_*] = z_*$. The following lemma is valid.

Lemma 3.2. Let Condition 2.1 be fulfilled. Then, for a number $\alpha > 0$ we can find a number $\eta > 0$ such that for each vector $v^* \in Q$ we can find a vector $u^* = u^*(t_*, y_*, z_*, v^*)$ $(u^* \in P)$ which for the motions $y_\Delta[t] = y_\Delta[t; t_*, y_*, u^*]$, $z_\Delta[t] = z_\Delta[t; t_*, z_*, v^*]$ ensures the estimate

$$\varepsilon(t_* + \Delta, y_\Delta[t_* + \Delta], z_\Delta[t_* + \Delta]) \leqslant \varepsilon(t_*, y_*, z_*) + \alpha\Delta \tag{3.4}$$

if only $\Delta \leqslant \eta$. Moreover, estimate (3.4) may be made uniform with respect to all $\{t_*, y_*, z_*\} \in \Gamma_\beta$ from (3.1).

Indeed, for the number $\alpha > 0$ we can find a number $\delta > 0$ such that the inequlities

$$|\psi(t_1, u, v, l) - \psi(t_2, u, v, l)| \leqslant \alpha / 2 \tag{3.5}$$

$$\max_{v \in Q} \min_{u \in P} \max_{l \in L} \psi(t_*, u, v, l) \leqslant \alpha / 2 \tag{3.6}$$

are valid if only $L = L(t_*, y_*, z_*, \delta)$ of (3.2), $|t_2 - t_1| \leqslant \delta$, $\|l\| = 1$, $u \in P$, $v \in Q$, $t_1, t_2 \in [t_0, \vartheta]$. We note that inequality (3.6) is fulfilled uniformly with respect to all positions $\{t_*, y_*, z_*\} \in \Gamma_\beta$.

For a known number δ we can choose a number η $(\eta \leqslant \delta)$ such that the estimates

$$\|y_\Delta[t]-y_*\|\leqslant\delta,\qquad \|z_\Delta[t]-z_*\|\leqslant\delta \tag{3.7}$$

are valid for any motions $y_\Delta[t]=y_\Delta[t;t_*,y_*,u]$, $z_\Delta[t]=z_\Delta[t;t_*,z_*,v]$ where $u\in P$, $v\in Q$, if only $|t-t_*|\leqslant\eta$, $\{t_*,y_*,z_*\}\in\Gamma_\beta$. We fix an arbitrary vector $v^*\in Q$ and at the position $\{t_*,y_*,z_*\}$ we define the vector $u^*=u^*(t_*,y_*,z_*,v^*)$ $(u^*\in P)$ from the minimum condition

$$\max_{l\in L}\psi(t_*,u^*,v^*,l)=\min_{u\in P}\max_{l\in L}\psi(t_*,u,v^*,l) \tag{3.8}$$

Here $L=L(t_*,y_*,z_*,\delta)$ is from (3.2), (3.5), (3.6). Comparing (3.8) and (3.6) we have

$$\max\nolimits_{l\in L}\psi(t_*,u^*,v^*,l)\leqslant\alpha/2 \tag{3.9}$$

We now compute the total derivative of the function $\varphi[t,l]=\varphi(t;y_\Delta[t],z_\Delta[t],l)$ $(\|l\|=1)$ in (2.1) along the motions $y_\Delta[t]=y_\Delta[t;t_*,y_*,u^*]$, $z_\Delta[t]=z_\Delta[t;t_*,z_*,v^*]$ on the interval $[t_*,t_*+\Delta]$ ($\Delta\leqslant\eta$ of (3.7)). Using (2.7), (3.5) we obtain

$$\frac{d\varphi[t,l]}{dt}=\psi(t,u^*,v^*,l)$$

$$\frac{d\varphi[t,l]}{dt}\leqslant\psi(t_*,u^*,v^*,l)+\frac{\alpha}{2} \tag{3.10}$$

From (3.10), (3.9) follow, in their own turn, the estimates

$$d\varphi[t,l]/dt\leqslant\alpha,\qquad \varphi[t,l]=\varphi(t,y_\Delta[t],z_\Delta[t],l)$$
$$\varphi[t_*+\Delta,l]\leqslant\varphi[t_*,l]+\alpha\Delta \tag{3.11}$$

if the vector $l\in L(t_*,y_*,z_*,\delta)$ of (3.2), (3.5), (3.6). Since we have considered unit vectors l, from (3.11), (2.3), (3.2) follows the inequality

$$\max_{l\in L}\varphi(t_*+\Delta,y_\Delta[t_*+\Delta],z_\Delta[t_*+\Delta],l)\leqslant\varepsilon(t_*,y_*,z_*)+\alpha\Delta \tag{3.12}$$

By virtue of the choice of number η $(\eta\leqslant\delta)$, estimates (3.7) are valid for the motions $y_\Delta[t]=y_\Delta[t;t_*,y_*,u^*]$, $z_\Delta[t]=z_\Delta[t;t_*,z_*,v^*]$. Therefore, from (3.12), as well as from (2.3), (2.1), (3.2), we have

$$\varepsilon(t_*+\Delta,y_\Delta[t_*+\Delta],z_\Delta[t_*+\Delta])\leqslant\varepsilon(t_*,y_*,z_*)+\alpha\Delta \tag{3.13}$$

Inequality (3.13) proves the lemma.

The definition of a u-stable system of sets was given in [5, 7]. We present this definition in a form suitable for what is to follow.

Definition 3.1. In the phase space $\{y,z\}$ let there be given a closed set G and a system of nonempty closed sets $W(t,\vartheta)(t_0\leqslant t\leqslant\vartheta)$, where $W(\vartheta,\vartheta)=G$. The system of sets $W(t,\vartheta)$ $(t_0\leqslant t\leqslant\vartheta)$ is said to be u-stable relative to G if, whatever be $t_*\in[t_0,\vartheta]$, $\{y_*,z_*\}\in W(t_*,\vartheta)$, $\eta\in[0,\vartheta-t_*)$, for any constant control $v^*[t]=v^*\in Q$ we can find a measurable control $u^*[\cdot]=u^*[t]\in P$ $(t_*\leqslant t\leqslant t_*+\eta)$ such that the inclusion

$$\{y_*[t_*+\eta],z_*[t_*+\eta]\}\in W(t_*+\eta,\vartheta)$$

is valid for the motions $y_*[t]=y_*[t;t_*,y_*,u^*[\cdot]]$, $z_*[t]=z_*[t;t_*,z_*,v^*]$.

Let us assume that the inequality

$$\varepsilon(t_0,y_0,z_0)>0 \tag{3.14}$$

is fulfilled at the initial game position $\{t_0,y_0,z_0\}$. In the space of vectors $\{y,z\}$

we specify the sets $W_\varepsilon(t, \vartheta)(t_0 \leqslant t \leqslant \vartheta)$ by the following relation:

$$W_\varepsilon(t, \vartheta) = \{y, z : \varepsilon(t, y, z) \leqslant \varepsilon\} \tag{3.15}$$

where $\varepsilon = \varepsilon(t_0, y_0, z_0)$. Since the function $\varepsilon(t, y, z)$ is continuous in all arguments., it is obvious that each of the sets $W_\varepsilon(t, \vartheta)$ is closed. Furthermore, taking inequality (3.14) into account, it can be shown (for example, see [5]) that the sets $W_\varepsilon(t, \vartheta)$ $(\varepsilon = \varepsilon(t_0, y_0, z_0))$ are not empty. The next assertion follows from Lemma 3.2.

Lemma 3.3. If Condition 2.1 is fulfilled, the system of sets $W_\varepsilon(t, \vartheta)$ is u-stable relative to $M_\varepsilon = \{q : \rho(\{q\}_m, M) \leqslant \varepsilon\}$.

We now define the first player's strategy $U_e = U_e(t, y, z)$ in the following manner [7]. If $\{y, z\} \in W_\varepsilon(t, \vartheta)$, we set $U_e(t, y, z) = P$. However, if $\{y, z\}$ does not belong to set $W_\varepsilon(t, \vartheta)$ of (3.15), then in the set $W_\varepsilon(t, \vartheta)$ we pick out all vectors w_0 nearest to $\{y, z\}$, i.e.

$$\rho(\{y, z\}, w_0) = \rho(\{y, z\}, W_\varepsilon(t, \vartheta))$$

As $U_e(t, y, z)$ we select all vectors u_e each of which satisfies the maximum condition

$$s' B^{(1)}(t) u_e = \max_{u \in P} s' B^{(1)}(t) u \tag{3.16}$$

for at least one value of $s = x - w^\circ$. Using the results of [7], we can prove the following assertion with the aid of Lemma 3.3.

Theorem 3.1. Let Condition 2.1 be valid. If the initial game position $\{t_0, y_0, z_0\}$ is such that inequality (3.14) is fulfilled, then strategy U_e of (3.16) ensures the estimate

$$\varepsilon(\vartheta; y[\vartheta], z[\vartheta]) \leqslant \varepsilon(t_0, y_0, z_0)$$

for any motions of systems (1.1),(1.2): $y[t] = y[t; t_0, y_0, U_e]$, $z[t] = z[t; t_0, z_0, v[\cdot]]$.

Condition 2.1 is always fulfilled if each of the sets $L_0 = L_0(t, y, z)$ in (2.5) consists of a single vector, which corresponds to the regular case of a game, examined in [5].

4. Under certain natural assumptions on the smoothness of the systems (1.1),(1.2) we prove that Condition 2.1 is also necessary for the function $\varepsilon(t, y, z)$ in the region $\varepsilon(t, y, z) > 0$ to be the value of the game in Problem 1.1. Let us assume that the matrices $Y[\vartheta, t] B^{(1)}(t)$, $Z[\vartheta, t] B^{(2)}(t)$ satisfy a Lipschitz condition in t

$$\| Y[\vartheta, t_1] B^{(1)}(t_1) - Y[\vartheta, t_2] B^{(1)}(t_2) \| \leqslant R_1 |t_1 - t_2| \tag{4.1}$$

$$\| Z[\vartheta, t_1] B^{(2)}(t_1) - Z[\vartheta, t_2] B^{(2)}(t_2) \| \leqslant R_2 |t_1 - t_2| \tag{4.2}$$

Here the norm of a matrix $C = \{c_{ij}\}$ $(i = 1, \ldots, n)$ $(j = 1, \ldots, k)$ is specified by the relation

$$\|C\| = \max_j \left[\sum_{i=1}^{n} c_{ij}^2\right]^{1/2}$$

The following theorem is valid.

Theorem 4.1. If the function $\varepsilon(t, y, z)$ in the region $\varepsilon(t, y, z) > 0$ is the value of the game, then Condition 2.1 is fulfilled.

Proof. We assume the contrary, i.e. Condition 2.1 is violated at some point $\{t_*, y_*, z_*\}$ where $\varepsilon(t_*, y_*, z_*) > 0$. Then we can find a vector $v^* \in Q$ such that the inequality

$$\max_{l_0\in L_0} \psi(t_*, u, v^*, l_0) \geqslant \alpha_0 > 0 \tag{4.3}$$

where $L_0 = L_0(t_*, y_*, z_*)$ and the function $\psi(t, u, v, l)$ is defined by formula (2.7), is valid for all vectors $u \in P$. By virtue of inequalities (4.1), (4.2) it is obvious that the function $\psi(t, u, v, l)$ also satisfies a Lipschitz condition in t

$$|\psi(t_1, u, v, l) - \psi(t_2, u, v, l)| \leqslant R\,|t_1 - t_2| \tag{4.4}$$

Let us arbitrarily specify the first player's position strategy $U = U(t, y, z)$. We show that the inequality

$$\varepsilon(t_* + \delta_0, y[t_* + \delta_0], z[t_* + \delta_0]) \geqslant \varepsilon(t_*, y_*, z_*) + {}^1/_4\, \alpha_0^2/R$$

$$(\delta_0 = {}^1/_2\alpha_0/R)$$

where the quantity α_0 is found from relation (4.3), is fulfilled for any of motions $y[t] = y[t; t_*, y_*, U]$ and for the motion $z[t] = z[t; t_*, z_*, v^*]$, generated by strategy U and by the control $v[t] = v^*$ from (4.3). To do this we examine the sequence of approximate motions $y_{\Delta_k}[t]$ of (1.5), converging uniformly to $y[t]$. By computing the total derivative of the function $\varphi[t, l] = \varphi(t, y_{\Delta_k}[t], z_{\Delta_k}[t], l)$ and using (4.4), we have

$$d\varphi[t, l]/dt \geqslant \psi(t_*, u[\tau_i^k], v^*, l) - R\delta_0 \quad (i = 0, 1, \ldots, p)$$

$$t \in [\tau_i^k, \tau_{i+1}^k], \quad \tau_{i+1}^k - \tau_i^k = \Delta_i^k, \quad \sum_{i=1}^{p} \Delta_i^k = \delta_0$$

$$\varphi[\tau_{i+1}^k, l] - \varphi[\tau_i^k, l] \geqslant \psi(t_*, u[\tau_i^k], v^*, l)\,\Delta_i^k - R\delta_0\Delta_i^k \tag{4.5}$$

From formula (4.5) there follows, in an obvious way, the inequality

$$\varphi[t_* + \delta_0 l] \geqslant \varphi[t_*, l] + \sum_{i=1}^{p} \psi(t_*, u[\tau_i^k], v^* l)\,\Delta_i^k - R\delta_0^2 \tag{4.6}$$

Using the fact that set P is convex, we write the following equality:

$$\psi(t_*, u^*, v^*, l)\,\delta_0 = \sum_{i=1}^{p} \psi(t_*, u[\tau_i^k], v^*, l)\,\Delta_i^k \tag{4.7}$$

$$\left(u^* = \frac{1}{\delta_0} \sum_{i=1}^{p} u[\tau_i^k]\,\Delta_i^k \in P\right)$$

We now choose a vector $l_* \in L_0(t_*, y_*, z_*)$ satisfying the relation

$$\psi(t_*, u^*, v^*, l_*) = \max_{l_0\in L_0} \psi(t_*, u^*, v^*, l_0) \geqslant \alpha_0 \tag{4.8}$$

From formulas (4.6), (4.7), (4.8) follows the estimate

$$\varphi[t_* + \delta_0, l_*] \geqslant \varphi[t_*, l_*] + \alpha_0\delta_0 - R\delta_0^2$$

Since the vector $l_* \in L_0(t_*, y_*, z_*)$, from this we obtain the obvious inequalities

$$\varphi[t_* + \delta_0, l_*] \geqslant \varepsilon(t_*, y_*, z_*) + \alpha_0\delta_0 - R\delta_0^2$$

$$\varepsilon(t_* + \delta_0, y_{\Delta_k}[t_* + \delta_0], z_{\Delta_k}[t_* + \delta_0]) \geqslant \varepsilon(t_*, y_*, z_*) - \alpha_0/2\delta_0$$

This proves Theorem 4.1.

The next assertion follows from Theorems 3.4, 4.1.

Theorem 4.2. In order for the function $\varepsilon\ (t,\ y,\ z)$ of (2.4) in the region $\varepsilon\ (t, y, z) > 0$ to be a value of the game, solving Problem 1.1, it is necessary and sufficient that Condition 2.1 hold.

We say that strategy $U = U\ (t,\ y,\ z)$ ensures the encounter of systems (1.1), (1.2) with the set M at the instant ϑ, if for any motions $y\ [t] = y\ [t;\ t_0,\ y_0,\ U],\ z\ [t] = z\ [t;\ t_0,\ z_0,\ v\ [\cdot]]$ the inclusion

$$\{z\ [t_*] - y\ [t_*]\}_m \in M$$

is valid for at least one value of $t_* = t_*\ (y\ [\cdot],\ z\ [\cdot])\ (t_0 \leqslant t_* \leqslant \vartheta)$. By $\vartheta_M = \vartheta_M\ (t_0,\ y_0,\ z_0)$ we denote the first instant at which the equality

$$\varepsilon\,(t_0, y_0, z_0, \vartheta_M) = 0 \tag{4.9}$$

is fulfilled, where the quantity $\varepsilon\ (t,\ y,\ z,\ \vartheta)$ was defined by relations (2.1)–(2.4) for an arbitrary instant ϑ. We can prove the next theorem by using the results of Lemma 3.1.

Theorem 4.3. Let $\vartheta_M = \vartheta_M\ (t_0,\ y_0,\ z_0)$ be the finite instant at which relation (4.9) is fulfilled for the first time. If Condition 2.1 is valid for the instant ϑ_M, there exists the first player's strategy U ensuring the encounter of systems (1.1), (1.2) with set M at the instant ϑ_M.

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